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GAUGE EQUIVALENCE BETWEEN TWO-DIMENSIONAL HEISENBERG FERROMAGNETS WITH SINGLE-SITE ANISOTROPY AND ZAKHAROV EQUATIONS[1]

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Abstract

Gauge equivalence between the two-dimensional continuous classical Heisenberg ferromagnets(CCHF) of spin $\frac{1}{2}$-the M-I equation with single-side anisotropy and the Zakharov equation(ZE) is established for the easy axis case. The anisotropic CCHF is shown to be gauge equivalent to the isotropic CCHF.

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In the study of 1+1 dimensional ferromagnets, a well known Lakshmanan and gauge equivalence take place between the continuous classical Heisenberg ferromagnets of spin $\frac{1}{2}$ and the nonlinear Schrodinger equations (e.g., Lakshmanan 1977, Zakharov and Takhtajan 1979, Nakamura and Sasaga 1982, Kundu and Pashaev 1983, Kotlyarov 1984). At the same time, there exit some integrable analogues of the CCHF in 2+1 dimensions(Ishimori 1982, Myrzakulov 1987). One integrable (2+1)-dimensional extension of the CCHF is the following Myrzakulov-I(M-I) equation with one-ion anisotropy

$$\mathbf{S}_t = (\mathbf{S} \wedge \mathbf{S}_y + u\mathbf{S})_x + v\mathbf{S} \wedge \mathbf{n}, \qquad (1a)$$

$$u_x = -\mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y), \qquad (1b)$$

$$v_x = \triangle(\mathbf{S}_y \cdot \mathbf{n}) \qquad (1c)$$

were the spin field $\mathbf{S} = (S_1, S_2, S_3)$ with the magnetude normalized to unity, u and v are scalar functions, $\mathbf{n} = (0, 0, 1)$, and $\triangle < 0$ and $\triangle > 0$ correspond respectively to the system with an easy plane and to that with an easy axis. Note that if the symmetry $\partial_x = \partial_y$ is imposed then the M-I equation (1) reduces to the well known Landau-Lifshitz equation with single-site anisotropy

$$\mathbf{S}_t = \mathbf{S} \wedge (\mathbf{S}_{xx} + \triangle(\mathbf{S} \cdot \mathbf{n})\mathbf{n}). \qquad (2)$$

The aim of this letter is the construction the (2+1)-dimensional nonlinear Schrodinger equation which is gauge equivalent to the M-I equation (or two-dimensional CCHF)(1) with the easy-axis anisotropy ($\triangle > 0$). Besides the gauge equivalence between anisotropic ($\triangle \neq 0$) and isotropic ($\triangle = 0$) CCNF (1) is established. The Lax representation of the M-I equation (1) may be given by (Myrzakulov 1987)

$$\psi_x = L_1\psi, \quad \psi_t = 2\lambda\psi_y + M_1\psi \qquad (3)$$

where

$$L_1 = i\lambda S + \mu[\sigma_3, S], \quad M_1 = 2\lambda A + 2i\mu[A, \sigma_3] + 4i\mu^2\{\sigma_3, V\}\sigma_3 \qquad (4)$$

with

$$S = \sum_{k=1}^{3} S_k \sigma_k, \quad V = \triangle \partial_x^{-1} S_y dx, \quad A = \frac{1}{4}([S, S_y]+2iuS), \ \mu = \sqrt{\frac{\triangle}{4}}, \triangle > 0.$$

Here σ_k is Pauli matrix, $[,]$ ($\{,\}$) denote commutator (anti-commutator) and λ is a spectral parameter. The matrix S has the following properties: $S^2 = I, \ S^* = S, \ trS = 0$. The compatibility condition of system (2) $\psi_{xt} = \psi_{tx}$ gives the M-I equation (1). Let us now consider the gauge transformation induced by $g(x, y, t) : \psi = g^{-1}\phi$, where $g^* = g^{-1} \in SU(2)$. It follows from the properties of the matrix S that it can be representies in the form $S = g^{-1}\sigma_3 g$. The new gauge equivalent operators L_2, M_2 there fore should be given by

$$L_2 = gL_1g^{-1} + g_xg^{-1}, \quad M_2 = gM_1g^{-1} + g_tg^{-1} - 2\lambda g_yg^{-1} \quad (5)$$

and satisfy the following system of equations

$$\phi_x = L_2\phi, \quad \phi_t = 2\lambda\phi_y + M_2\phi \quad (6)$$

Now choosing

$$g_xg^{-1} + \mu g[\sigma_3, S]g^{-1} = U_0, \ \ gSg^{-1} = \sigma_3, \quad (7a)$$

$$g_tg^{-1} + 2i\mu g[A, \sigma_3]g^{-1} + 4i\mu^2 g\{\sigma_3, V\}\sigma_3 g^{-1} = V_0, \quad (7b)$$

with

$$U_0 = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad V_0 = i\sigma_3(\partial_x^{-1}|q|_y^2 - U_{0y}).$$

where $q(x, y, t)$ the new complex valued fields. Hence we finally obtain

$$L_2 = i\lambda\sigma_3 + U_0, \quad M_2 = V_0 \quad (8)$$

The compatibility condition $\phi_{xt} = \phi_{tx}$ of the system (6) with the operators L_2, M_2 (8) leads to the (2+1)-dimensional Zakharov equation (Zakharov 1979, Strachan 1993)

$$iq_t = q_{xy} + wq, \quad w_x = 2(|q|^2)_y \quad (9)$$

We note that under the reduction $\partial_y = \partial_x$ equation(9) becomes the well known (1+1)-dimensional NLSE. Thus we have

shown that the M-I equation with single-site anisotropy (the two dimensional continuum Heisenberg ferromagnets) is gauge equivalent to the (2+1)-dimensional NLSE - the Zakharov equation(9). On the other hand, equations (9) is gauge and geometrical equivalent to the isotropic M-I equation (Myrzakulov 1987; Nugmanova 1992, Myrzakulov 1994)

$$iS'_t = \frac{1}{2}([S', S'_y] + 2iu'S')_x \tag{10a}$$

$$u'_x + \mathbf{S}'(\mathbf{S}'_x \wedge \mathbf{S}'_y) = 0 \tag{10b}$$

which is the compatibility condition of the following system

$$f_x = L'_1 f, \quad f_t = 2\lambda f_y + \lambda M'_1 f \tag{11}$$

where

$$L'_1 = i\lambda S', \quad M'_1 = \frac{1}{2}([S', S'_y] + 2iuS'). \tag{12}$$

Now we show below the gauge equivalence between anisotropic and isotropic M-I equations (1) and (10),respectively. Indeed the Lax representations (3) and (11), which reproduce equations (1) and (10) respectively can be obtained from each other by the λ-independent gauge transformation $h(x,y,t) = \psi^{-1}|_{\lambda=b}$ as

$$L'_1 = hL_1h^{-1} + h_xh^{-1}, \quad M'_1 = hM_1h^{-1} + h_th^{-1}. \tag{13}$$

In this way, the solutions of eq.(1) and (10) connected each other by formulas $S = h^{-1}S'h$. Now we present the important relations between field variables ψ and S

$$|\psi|^2 = \frac{1}{2}[\mathbf{S}_x^2 - 8\mu S_{3x} + 16\mu^2(1 - S_2^3)] \tag{14a}$$

$$\psi\bar{\psi}_x - \bar{\psi}\psi_x = \frac{i}{4}\mathbf{S}\cdot[\mathbf{S_{xx}} + 16\mu^2(\mathbf{S}\cdot\mathbf{n})\mathbf{n})] + 4\mu\mathbf{S}\cdot(\mathbf{S}_{xx}\wedge\mathbf{n}) \tag{14b}$$

These relations coincide with the corresponding connections between q and S of one-dimensional case(Guispel and Capel 1982, Nakamura and Sasada 1982).

Note that the two-dimensional CHSC with ($\triangle < 0$) easy plane single-site anisotropy is gauge equivalent to the some general (2+1)- dimensional NLSE (Myrzakulov 1987):

$$iq_t = q_{xy} + wq \qquad (15a)$$

$$ip_t = -p_{xy} - wp \qquad (15b)$$

$$w_x = 2(pq)_y \qquad (15c)$$

Besides, the two-dimensional CHSC, when $S \in SU(1,1)/U(1)$, i.e. the non-compact group, is gauge and Lakshmanan equivalent to the Zakharov equation(15) with the repulsive interaction, $p = -\bar{q}$ (Myrzakulov 1987).

Finally we note that the M-I equation(1) is the particular case of the M-III equation(Myrzakulov 1987)

$$\mathbf{S}_t = (\mathbf{S} \wedge \mathbf{S}_y + u\mathbf{S})_x + 2b(cb+d)\mathbf{S}_y - 4cv\mathbf{S}_x + \mathbf{S} \wedge \mathbf{V} \qquad (16a)$$

$$u_x = -\mathbf{S}\dot{(}\mathbf{S}_x \wedge \mathbf{S}_y), \qquad (16b)$$

$$v_x = \frac{1}{4(2bc+d)^2}(\mathbf{S}^2_{1x})_y \qquad (16c)$$

$$\mathbf{V}_x = J\mathbf{S}_y, \qquad (16d)$$

where J is a anisotropic matrix. These equations admit the some integrable reductions.

To summarise, we have established the gauge equivalence between the classical continuum Heisenberg spin $\frac{1}{2}$ chain with easy axis ($\triangle > 0$) anisotropy and the nonlinear Schrodinger equation of the attractive type in two-dimensions. In additional the anisotropic CHSC(1) is shown to be gauge equivalent to isotropic CHSC(10). These results corresponds to the physical statements on thermodynamical equivalence between the corresponding quantum versions of Heisenberg ferromagnet(antiferromagnet) and Bose gas with attractive(repulsive) interaction, respectively(Yang and Yang 1966).

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